7. Morgenstern, D., Mathematische Begründung der Scheibentheorie. Arch. Ration Mech. and Analysis, Vol. 3, №1, 1959.
8. Morgenstern, D., Herleitung der Plattentheorie aus der dreidimensionalen Elastizitätstheorie. Arch. Ration. Mech. and Analysis, Vol. 2, №2, 1959.
9. Morgenstern, D., Bernoullische Hypothesen bei Balken und Platten Theorie. Z. angew. Math. und Mech., Bd. 39, №№9 – 11, 1959.
10. Gussein-Zade, M. I., On the derivation of a theory of bending of layered plates. PMM Vol. 32, №2, 1968.
11. Gussein-Zade, M. I., The state of stress in the boundary layer for layered plates. In: Proc. of the Seventh All-Union Conf. on the Theory of Plates and Shells. Moscow, "Nauka", 1970.
12. Agalovian, L. A., On the equations of the bending of anisotropic plates. Proc. of the Seventh All-Union Conf. on the Theory of Plates and Shells. Moscow, "Nauka", 1970.
13. Friedrichs, K. O., On the boundary value problems of theory of elasticity and Korn's inequality. Ann. Math., Vol. 48, №2, 1947.
14. Mikhlin, S. G., The Problem of the Minimum of a Quadratic Functional. (Translation from Russian), San Francisco, Holden-Day, 1965.
15. Duvaut, G. and Lions, J. L., Les inéquations en méchanique et en physique. Paris, Dunod, 1972.
16. Gobert, J., Une inéquation fondamentale en théorie d'élasticite. Bull. Soc. roy. sci. Liège, Vol. 31, №№ 3, 4, 1962.
17 Mosolov, P. P. and Miasnikov, V. P., The proof of Korn's inequality. Dokl. Akad. Nauk SSSR, Vol. 201, №1, 1971.
18. Vainberg, M. M., The Variational Method and the Method of Monotone Operators in the Theory of Nonlinear Equations. Moscow, "Nauka", 1972.
19. Sobolev, S. L., Applications of Functional Analysis in Mathematical Physics. (Translation from Russian), Providence, American Mathematical Society, 1963.
20. Akhiezer, N. I. and Glazman, I. M., Theory of Linear Operators in Hilbert Space, New York, Ungar, 1961.

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CONTACT PROBLEM OF ROLLING OF A VISCOELASTIC CYLINDER ON A BASE OF THE SAME MATERIAL

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The problem of rolling of a viscoelastic cylinder on a base of the same material is solved under the assumption that the whole contact area consists of two sections: a section with adhesion and a section with slipping of the contacting surfaces. Equations are found to determine the length of the contact area and the adhesion section, as are expressions for the stresses on the contact area.

Reynolds [1] noted that relative sliding of the contacting surfaces occurs during the rolling of elastic solids because of their deformation. The results of experiments [2, 3] verify the importance of the sliding friction forces at the contact which originate in connection with the differences in the curvatures of the surfaces making contact. In many cases the imperfect elasticity of the real materials [4] is no less important. As the cylinder rolls on a base of viscoelastic material, the deformations of the material ahead of and behind the moving cylinder are different, which is due to the presence of an aftereffect. Consequently, a shift of the contact zone occurs in the direction of motion and a nonsymmetric pressure distribution of the cylinder on the base takes place over the contact area. As a result a rolling friction moment originates.

The simultaneous effect of the above-mentioned causes affecting the resistance to rolling is considered below.

The relations between the strain and stress components in an isotropic viscoelastic body are taken in the following form:

$$\varepsilon_{x^\circ} + \alpha \frac{\partial \varepsilon_{x^\circ}}{\partial t} = \frac{1}{E}\left(\sigma_{x^\circ} + \beta \frac{\partial \sigma_{x^\circ}}{\partial t}\right) - \frac{\nu}{E}\left(\sigma_{y^\circ} + \beta \frac{\partial \sigma_{y^\circ}}{\partial t}\right)$$

$$\varepsilon_{y^\circ} + \alpha \frac{\partial \varepsilon_{y^\circ}}{\partial t} = \frac{1}{E}\left(\sigma_{y^\circ} + \beta \frac{\partial \sigma_{y^\circ}}{\partial t}\right) - \frac{\nu}{E}\left(\sigma_{x^\circ} + \beta \frac{\partial \sigma_{x^\circ}}{\partial t}\right)$$

$$\gamma_{x^\circ y^\circ} + \alpha \frac{\partial \gamma_{x^\circ y^\circ}}{\partial t} = \frac{2(1+\nu)}{E}\left(\tau_{x^\circ y^\circ} + \beta \frac{\partial \tau_{x^\circ y^\circ}}{\partial t}\right)$$

Here α and β are parameters characterizing the viscous properties of the medium ($\alpha > \beta$).

Let a viscoelastic cylinder move over a viscoelastic base with a velocity w which is much smaller than the speed of sound in the viscoelastic body, which permits inertial terms to be neglected in the equilibrium equations. Considering the radius of curvature of the cylinder R to be large compared to the dimensions of the contact area, let us replace the cylinder by an upper half-plane. Let us examine the problem of the contact of two half-planes, where the area of contact shifts with velocity w along them.

Let us introduce the moving coordinate system

$$x = x^\circ - wt, \qquad y = y^\circ$$

As the cylinder moves uniformly, the motion of the medium can be considered steady relative to the coordinate system moving translationally together with the center of the cylinder. Then the displacements and stresses do not depend explicitly on time and are functions of only the coordinates. Let us use the notation

$$\varepsilon_{ij} + \alpha \frac{\partial \varepsilon_{ij}}{\partial t} = \varepsilon_{ij} - \alpha w \frac{\partial \varepsilon_{ij}}{\partial x} = \varepsilon_{ij}^*(x, y) \tag{1}$$

$$\sigma_{ij} + \beta \frac{\partial \sigma_{ij}}{\partial t} = \sigma_{ij} - \beta w \frac{\partial \sigma_{ij}}{\partial x} = \sigma_{ij}^*(x, y)$$

$$u - \alpha w \frac{\partial u}{\partial x} = u^*(x, y)$$

$$v - \alpha w \frac{\partial v}{\partial x} = v^*(x, y)$$

Equations equivalent to the equilibrium, strain compatibility and Hooke's law equations for an isotropic elastic body are satisfied for the quantities ε_x^*, ε_y^*, γ_{xy}^*, σ_x^*, σ_y^*, τ_{xy}^* introduced in this manner.

Let us consider that the whole contact area consists of two sections: sliding of the contacting surfaces occurs on one $(-a, c)$, while adhesion holds on the other (c, b). Because of the smallness of the strain the boundary conditions on the surface will refer to the undeformed state of the medium $(y = 0)$. The relationship [5]

$$\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x}\right)_{y=0} = -f'(x) \tag{2}$$

holds between the normal displacements on the contact area. Here v_1 and v_2 are the normal cylinder and half-plane displacements, respectively, and $f(x)$ is the equation of the cylinder outline prior to deformation. We take $f(x) = x^2 / 2R$ because of the smallness of the deformation. Moreover, a linear relationship between the normal pressure σ_y and the tangential stress resultants τ_{xy} acting in the lower half-plane

$$\tau_{xy} + \rho\sigma_y = 0$$

holds on the sliding section $(-a, c)$, where ρ is the coefficient of sliding friction.

The velocities of the horizontal displacements of points of the cylinder $w - \omega R + du_1 / dt$ (ω is the angular velocity of the cylinder) and of points of the half-plane du_2 / dt are equal on the adhesion section (c, b). In the coordinate system coupled to the cylinder, the condition of equal velocities is written as follows:

$$\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\right)_{y=0} = \delta, \qquad \delta = \frac{\omega R - w}{w}$$

The normal pressure P equilibrating the weight per unit length of the cylinder and the rotational moment M act on the moving cylinder. The surface of the viscoelastic body is free of stress resultants outside the contact area.

Let us introduce two functions of a complex variable $w_1(z)$ and $w_2(z)$ in the lower half-plane, which are Cauchy type integrals whose densities are, respectively, equal to the magnitudes of the normal pressure and the shear stress resultants acting on the half-plane boundary

$$w_1(z) = U_1 - iV_1 = \int_{-a}^{b} (\sigma_y^*)_{y=0} \frac{dt}{t-z}$$

$$w_2(z) = U_2 - iV_2 = \int_{-a}^{b} (\tau_{xy}^*)_{y=0} \frac{dt}{t-z}$$

where at infinity

$$w_1(z) \sim -\frac{P}{z} + o\left(\frac{1}{|z|}\right), \qquad w_2(z) \sim -\frac{Q}{z} + o\left(\frac{1}{|z|}\right)$$

$$P = \int_{-a}^{b} (\sigma_y)_{y=0}\, dx, \qquad Q = \int_{-a}^{b} (\tau_{xy})_{y=0}\, dx$$

It has here been taken into account that the stresses on the boundary of the contact area should equal zero at the points $-a$ and b because of smoothness of the roller contour.

Expressing the images of the pressure $(\sigma_y^*)_{y=0}$, the shear stress resultants $(\tau^*_{xy})_{y=0}$, the derivatives of the displacements $(\partial u_1^* / dx)_{y=0}$, $(\partial u_2^* / \partial x)_{y=0}$, $(\partial v_1^* / \partial x)_{y=0}$, $(\partial v_2^*/\partial x)_{y=0}$ in terms of the real and imaginary parts of the functions $w_1(z)$ and $w_2(z)$ [5] and substituting them into the boundary conditions, modified somewhat, taking account of (1), we obtain a conjugate problem to seek two functions $w_1(z)$ and $w_2(z)$ analytic in the lower half-plane

$$V_1 = 0, \quad V_2 = 0, \quad -\infty < x < -a, \ b < x < +\infty \tag{3}$$

$$U_1 = -(x - \alpha w)/KR, \quad V_2 = -\rho V_1, \quad -a < x < c$$

$$U_1 = -(x - \alpha w)/KR, \quad U_2 = \delta / K, \quad c < x < b$$

$$(K = 4(1 - \nu^2)/\pi E)$$

Here E is Young modulus, and ν is the Poisson's ratio of the cylinder and base materials.

The functions solving the problem of a linear conjugate with the listed conditions (3) are

$$w_1(z) = -\frac{1}{\sqrt{(z+a)(z-b)}} \int_{-a}^{b} \frac{x - \alpha w}{\pi K R} \sqrt{(a+x)(b-x)} \frac{dx}{x - z} - \tag{4}$$

$$\frac{C_1}{\sqrt{(z+a)(z-b)}} = U_1 - iV_1$$

$$w_2(z) = -\sqrt{\frac{z-c}{z-b}} \left[\int_{-a}^{c} \frac{\rho}{\pi} (V_1)_{y=0} \sqrt{\frac{b-x}{c-x}} \frac{dx}{x-z} - \right.$$

$$\left. \frac{\delta}{\pi K} \int_{c}^{b} \sqrt{\frac{b-x}{x-c}} \frac{dx}{x-z} \right] - \frac{C_2}{\sqrt{(z-c)(z-b)}} = U_2 - iV_2$$

Here C_1 and C_2 are some constants with which the solutions of the homogeneous conjugate problem enter. It again follows from the conditions at infinity that $C_1 = P$. The functions found on the real axis possess integrable singularities, but this circumstance does not involve irregularity in the stress components.

The representations of the stresses in the contact area are expressed in terms of the imaginary parts of the functions $w_1(z)$ and $w_2(z)$ by means of the formulas

$$(\sigma_y^*)_{y=0} = \frac{1}{\pi} (V_1)_{y=0}, \qquad (\tau_{xy}^*)_{y=0} = \frac{1}{\pi} (V_2)_{y=0}$$

Separating the imaginary parts in (4) for $z = x - i0$, we obtain for $-a < x < b$

$$(V_1)_{y=0} = \frac{1}{\sqrt{(a+x)(b-x)}} \int_{-a}^{b} \frac{t - \alpha w}{\pi K R} \sqrt{(a+t)(b-t)} \frac{dt}{t - x} + \tag{5}$$

$$\frac{P}{\sqrt{(a+x)(b-x)}}$$

on the sliding section $-a < x < c$

$$(V_2)_{y=0} = -\rho (V_1)_{y=0} \tag{6}$$

and on the adhesion section $c < x < b$

$$(V_2)_{y=0} = \sqrt{\frac{x-c}{b-x}} \left[\int_{-a}^{c} \frac{\rho}{\pi} V_1(t)|_{y=0} \sqrt{\frac{b-t}{c-t}} \frac{dt}{t-x} - \right.$$

$$\left. \frac{\delta}{\pi K} \int_{c}^{b} \sqrt{\frac{b-t}{t-c}} \frac{dt}{t-x} \right] + \frac{C_2}{\sqrt{(x-c)(b-x)}}$$

Evaluating the integral

$$\int_{-a}^{b} \frac{t-\alpha w}{\pi K R} \sqrt{(a+t)(b-t)} \frac{dt}{t-x} = \frac{(a+b)^2}{8KR} + \frac{(x-\alpha w)(b-a-2x)}{2KR} - \tag{7}$$

$$\begin{cases} \dfrac{x-\alpha w}{KR}(b-x)\left|\dfrac{a+x}{b-x}\right|^{1/2} & -\infty < x < -a,\ b < x < +\infty \\ 0, \quad -a < x < b \end{cases}$$

for the function $(V_1)_{y=0}$ in (5) at $-a < x < b$ we obtain the following expression:

$$(V_1)_{y=0} = \frac{1}{\sqrt{(a+x)(b-x)}}\left[\frac{(a+b)^2}{8KR} + \frac{(x-\alpha w)(b-a-2x)}{2KR} + P\right] = \pi(\sigma_y{}^*)_{y=0} \tag{8}$$

Let us introduce the notation

$$F(x) = \frac{(a+b)^2}{8KR} + \frac{(x-\alpha w)(b-a-2x)}{2KR} + P \tag{9}$$

and let us calculate the value of the integrals

$$\int_{-a}^{c} V_1(t)\,|_{y=0} \sqrt{\frac{b-t}{c-t}} \frac{dt}{t-x} = \frac{\pi}{2KR}(b-c-2x+2\alpha w) + \tag{10}$$

$$\begin{cases} \dfrac{\pi F(x)}{c-x}\left|\dfrac{c-x}{a+x}\right|^{1/2}, & -\infty < x < -a, \quad c < x < +\infty \\ 0, \quad -a < x < c \end{cases}$$

$$\int_{c}^{b} \sqrt{\frac{b-t}{t-c}} \frac{dt}{t-x} = \begin{cases} -\pi\left[1-\left|\dfrac{b-x}{c-x}\right|^{1/2}\right], & -\infty < x < c,\ b < x < +\infty \\ -\pi \quad c < x < b \end{cases}$$

Then on the adhesion section

$$(V_2)_{y=0} = -\rho(V_1)_{y=0} + \left[\frac{\rho}{2KR}(b-c-2x+2\alpha w) + \frac{\delta}{K}\right] \times \tag{11}$$

$$\sqrt{\frac{x-c}{b-x}} + \frac{C_2}{\sqrt{(x-c)(b-x)}} = \pi(\tau_{xy}^*)_{y=0}$$

Knowing the expression for the representations of the stresses (8) and (11), we use (1) and find the true stresses at any point on the contact area, for $-a < x < b$

$$(\sigma_y)_{y=0} = -\frac{1}{\pi\beta w}\exp\left(\frac{x}{\beta w}\right)\int_{-a}^{x}\left[\frac{(a+b)^2}{8KR} + \frac{(t-\alpha w)(b-a-2t)}{2KR} + P\right] \times \tag{12}$$

$$\frac{\exp\left(-\dfrac{t}{\beta w}\right)dt}{\sqrt{(a+t)(b-t)}}$$

on the slip section $-a < x < c$

$$(\tau_{xy})_{y=0} = -\rho(\sigma_y)_{y=0}$$

and on the adhesion section $c < x < b$

$$(\tau_{xy})_{y=0} = -\rho(\sigma_y)_{y=0} - \frac{1}{\pi\beta w}\exp\left(\frac{x}{\beta w}\right)\int_{c}^{x}\left[\frac{\rho}{2KR}(b-c-2t+2\alpha w) + \right. \tag{13}$$

$$\frac{\delta}{K}\Big]\sqrt{\frac{t-c}{b-t}}\exp\left(-\frac{t}{\beta w}\right)dt-\frac{C_2}{\pi\beta w}\exp\left(\frac{x}{\beta w}\right)\int_c^x\frac{\exp\left(-\frac{t}{\beta w}\right)dt}{\sqrt{(t-c)(b-t)}}$$

Four unknown constants — a, b, c, C_2 enter into these expressions. They can be determined by satisfying the boundary conditions for the real stresses and displacements in the viscoelastic body.

Determining $\partial v_1 / \partial x = v_{1x}'$ and $\partial v_2 / \partial x = v_{2x}'$ from the equations

$$-\frac{\partial v_{ix}'}{\partial x}+\frac{1}{\alpha w}v_{ix}'=\frac{1}{\alpha w}(v_{ix}')^* \qquad (i=1,2)$$

we have

$$(v_{1x}'+v_{2x}')_{y=0}=-\frac{1}{\alpha w}\exp\left(\frac{x}{\alpha w}\right)\int_{-\infty}^{x}(v_{1x}'+v_{2x}')^*_{y=0}\exp\left(-\frac{t}{\alpha w}\right)dt= \tag{14}$$

$$-\frac{1}{\alpha w}\exp\left(\frac{x}{\alpha w}\right)\int_{-\infty}^{x}K(U_1)_{y=0}\exp\left(-\frac{t}{\alpha w}\right)dt$$

The strains infinitely remote from the contact area vanish, hence the following condition should be satisfied:

$$\int_{-\infty}^{+\infty}(U_1)_{y=0}\exp\left(-\frac{t}{\alpha w}\right)dt=0 \tag{15}$$

Taking account of (2) and (15) we obtain the following equation from (14) for the point $x = b$

$$-\frac{\alpha w b}{KR}\exp\left(-\frac{b}{\alpha w}\right)=\int_{b}^{+\infty}(U_1)_{y=0}\exp\left(-\frac{t}{\alpha w}\right)dt \tag{16}$$

Extracting the real part in the function $w_1(z)$ in (4) in the interval $(b, +\infty)$ and using (7), (9) we find the form of the function $(U_1)_{y=0}$ for $b < x < +\infty$

$$(U_1)_{y=0}=-\frac{F(x)}{\sqrt{(x+a)(x-b)}}-\frac{x-\alpha w}{KR}$$

After having performed the two necessary calculations and manipulations, (16) becomes

$$\left[P-\frac{(a+b)^2}{8KR}\right]K_0\left(\frac{a+b}{2\alpha w}\right)-\frac{b^2-a^2}{4KR}K_1\left(\frac{a+b}{2aw}\right)=0 \tag{17}$$

where $K_0(x)$ and $K_1(x)$ are cylinder functions of imaginary argument.

The second equation to determine the boundaries of the contact area (the points $-a$ and b) is obtained from the condition that the normal pressure at the point b equals zero

$$\int_{-a}^{b}(V_1)_{y=0}\exp\left(-\frac{x}{\beta w}\right)dx=0$$

Substituting here the value of the function $(V_1)_{y=0}$ from (8) and integrating, we obtain an equation in which the Bessel functions of imaginary argument $I_0(x)$ and $I_1(x)$ enter

$$\left[P-\frac{(a+b)^2}{8KR}\right]I_0\left(\frac{a+b}{2\beta w}\right)+\left(\frac{b-a}{2}-\alpha w+\beta w\right)\frac{a+b}{2KR}I_1\left(\frac{a+b}{2\beta w}\right)=0 \tag{18}$$

Let us use the notation: $l = a + b$ equals the length of the contact area, and $d = (b - a) / 2$ is the coordinate of the middle of the contact area. From (17) and (18) we obtain an equation to determine the length of the contact area l

$$\left[P-\frac{l^2}{8KR}\right]\left[I_0\left(\frac{l}{2\beta w}\right)K_1\left(\frac{l}{2\alpha w}\right)+K_0\left(\frac{l}{2\alpha w}\right)I_1\left(\frac{l}{2\beta w}\right)\right]+ \\ w(\beta-\alpha)\frac{l}{2KR}I_1\left(\frac{l}{2\beta w}\right)K_1\left(\frac{l}{2\alpha w}\right)=0 \tag{19}$$

The coordinate of the middle of the area d is defined as follows:

$$d=\left[P-\frac{l^2}{8KR}\right]\frac{2KR}{l}K_0\left(\frac{l}{2\alpha w}\right)\Big/K_1\left(\frac{l}{2\alpha w}\right) \tag{20}$$

Let us find the values of the constants c and C_2. From the condition that the velocities on the adhesion section are equal we obtain

$$\int_b^{+\infty}(U_2)_{y=0}\exp\left(-\frac{x}{\alpha w}\right)dx=\frac{\delta\alpha w}{K}\exp\left(-\frac{b}{\alpha w}\right) \tag{21}$$

which is derived analogously to (16). From (14) and (10) we determine the form of the function $(U_2)_{y=0}$ for $b<x<+\infty$

$$(U_2)_{y=0}=-\left[\frac{\rho}{2KR}(b-c-2x+2\alpha w)+\frac{\delta}{K}\right]\sqrt{\frac{x-c}{x-b}}+\frac{\delta}{K}+ \\ \frac{\rho F(x)}{\sqrt{(x+a)((x-b)}}-\frac{C_2}{\sqrt{(x-c)(x-b)}}$$

It follows from (16) that

$$\int_b^{+\infty}F(x)\exp\left(-\frac{x}{\alpha w}\right)\frac{dx}{\sqrt{(x+a)(x-b)}}=0$$

Hence, after evaluating the integral in the left side of (21), we obtain

$$\left[C_2-\frac{\rho(b-c)(b+c-2\alpha w)}{4KR}+\frac{\delta(b-c)}{2K}\right]K_0\left(\frac{b-c}{2\alpha w}\right)- \\ \left[\frac{\rho(b^2-c^2)}{4KR}-\frac{\delta(b-c)}{2K}\right]K_1\left(\frac{b-c}{2\alpha w}\right)=0 \tag{22}$$

Finally, the requirement that the shear stress resultant on the boundary of the contact area be equal at the point $x=b$, results in the condition

$$\int_{-a}^{b}V_2(x)|_{y=0}\exp\left(-\frac{x}{\beta w}\right)dx=0$$

Substituting the expression for $(V_2)_{y=0}$ from (6), (11) here and performing the necessary calculations, we obtain the equation

$$\left[C_2-\frac{\rho(b-c)(b+c-2\alpha w)}{4KR}+\frac{\delta(b-c)}{2K}\right]I_0\left(\frac{b-c}{2\beta w}\right)+ \\ \left[\frac{\rho(b-c)(b+c-2\alpha w+2\beta w)}{4KR}-\frac{\delta(b-c)}{2K}\right]I_1\left(\frac{b-c}{2\beta w}\right)=0 \tag{23}$$

Let us denote the length of the adhesion section by $m=b-c$. Then taking account of the notation introduced earlier $b+c=2d+l-m$. From (22) and (23) we obtain an equation to determine the length of the adhesion section m

$$\frac{m}{2K}\left\{\left[\frac{\rho}{2R}(2d+l-m-2\alpha w+2\beta w)-\delta\right]I_1\left(\frac{m}{2\beta w}\right)K_0\left(\frac{m}{2\alpha w}\right)+\left[\frac{\rho}{2R}(2d+l-m)-\delta\right]I_0\left(\frac{m}{2\beta w}\right)K_1\left(\frac{m}{2\alpha w}\right)\right\}=0 \tag{24}$$

The constant C_2 is defined as follows:

$$C_2=\frac{m}{2K}\left\{\frac{\rho}{2R}(2d+l-m-2\alpha w)-\delta-\left[\frac{\rho}{2R}(2d+l-m-2\alpha w+2\beta w)-\delta\right]I_1\left(\frac{m}{2\beta w}\right)\Big/ I_0\left(\frac{m}{2\beta w}\right)\right\} \tag{25}$$

All the unknown constants of the problem have therefore been determined.

The length of the contact area l can be found from the solution of (19). In the majority of cases this equation is solved numerically [6]. The solution of (19) was carried out in the paper for values of the parameters α and β on the order of 10^{-1} sec up to 10^6 sec; the values 10 cm/sec, 100 cm/sec and 1000 cm/sec were taken for the velocity w. In all these cases the length of the contact area was close to $\sqrt{8KRP\beta/\alpha}$, which equals the length of the contact area during rolling over an elastic material with elastic modulus $H=\alpha E/\beta$. The modulus H is the instantaneous modulus of elasticity for the material under consideration. Knowing the length of the contact area, the coordinate of the middle of the area d can be determined by means of (20). It is hence seen that $d>0$, i.e. the area is shifted in the direction of the cylinder motion. The length of the adhesion section m is found from the solution of (24) and the constant C_2 from (25). If $\alpha=\beta$, then these equations yield the solution of the problem of rolling of an elastic cylinder over a base of the same material (with elastic modulus E). Let us use the relationship

$$I_0(x)K_1(x)+I_1(x)K_0(x)=x^{-1}$$

It then follows from (19) and (20) that the length of the contact area is $l=\sqrt{8KRP}$ and it is located symmetrically relative to the center of the cylinder ($d=0$). From (24) we obtain that in the case a section with adhesion ($m\neq 0$) exists, the section with slip has the length $l-m=2\delta R/\rho$. Substituting the values found for the unknown constants in (12) and (13), we obtain expressions for the pressures and shear stress resultants in the contact area of an elastic cylinder with an elastic base

$$(\sigma_y)_{y=0}=\frac{\sqrt{a^2-x^2}}{\pi KR},\quad -a<x<a$$

$$(\tau_{xy})_{y=0}=\begin{cases}-\dfrac{\rho}{\pi KR}\sqrt{a^2-x^2}, & -a<x<c\\[2mm] -\dfrac{\rho}{\pi KR}\sqrt{a^2-x^2}+\dfrac{\rho}{\pi KR}\sqrt{(a-x)(x-c)}, & c<x<a\end{cases}$$

$$(a=\sqrt{2KRP},\quad c=-\sqrt{2KRP}+2\delta R/\rho)$$

Let us determine the resistance of the cylinder to motion. The total shear stress on the contact area Q is calculated as follows:

$$Q=\int_{-a}^{b}(\tau_{xy})_{y=0}dx=\int_{-a}^{b}(\tau_{xy}^{*})_{y=0}dx=\int_{-a}^{b}\frac{1}{\pi}(V_2)_{y=0}dx \tag{26}$$

since by virtue of the continuity of the stresses on the boundary of the contact area

$$\int_{-a}^{b} \left(\frac{\partial \tau_{xy}}{\partial x}\right)_{y=0} dx = 0$$

The last integral in (26) is easily computed. Taking account of (25) and the notation introduced earlier, we obtain the following expression for Q

$$Q = -\rho P + \frac{\rho m^2}{8KR} - \frac{m}{2K}\left[\frac{\rho}{2R}(2d + l - m - 2\alpha w + 2\beta w) - \delta\right] \times I_1\left(\frac{m}{2\beta w}\right) \Big/ I_0\left(\frac{m}{2\beta w}\right) \qquad (27)$$

A cylinder resistance force Q^* equal in magnitude and opposite in direction to the force Q acts on the cylinder axis. Moreover, as the cylinder rolls over the viscoelastic base, the vertical components of the reaction of the viscoelastic medium does not pass through the cylinder center of gravity. Hence, a couple with a moment

$$M_1 = \int_{-a}^{b} x(\sigma_y)_{y=0} dx = \int_{-a}^{b} \frac{1}{\pi} x (V_1)_{y=0} dx - \beta w P \qquad (28)$$

resists the cylinder motion. It has here been taken into account that

$$\int_{-a}^{b} x \left(\frac{\partial \sigma_y}{\partial x}\right)_{y=0} dx = -P$$

because of the continuity of the stresses on the boundary of the contact area. Substituting the value of $(V_1)_{y=0}$ from (8) into (28), we obtain the following expression for the moment M_1:

$$M_1 = P(d - \beta w) + \frac{l^2}{8KR}(\alpha w - d)$$

The moment M_1, together with the moment of the force Q in (27) relative to the center of the cylinder $M_2 = QR$, produce a rolling friction moment $M^* = M_1 + M_2$. In order for uniform cylinder motion to hold, the moment M applied to it must be equal in absolute value to the moment M^* of the rolling friction.

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REFERENCES

1. Reynolds, O., On rolling friction. Phil. Trans. Roy. Soc., London, A, Vol. 166, pt. 1, 1876.
2. Pinegin, S. V. and Orlov, A. V., Resistence to motion for certain types of free rolling. Izv. Akad. Nauk SSSR, Mekh. i Mashinostr., №3, 1961.
3. Konvisarov, D. V. and Pokrovskaia, A. A., Influence of the radii of curvature of cylindrical bodies on their resistance to rerolling under different loads. Trudy Sib. Phys.-Tech. Inst., Tomsk, Vol. 34, 1955.
4. Tabor, R., The mechanism of rolling friction. Phil. Mag., Ser. 7, Vol. 43, №345, 1952.
5. Galin, L. A., Contact Problems of Elasticity Theory. Gostekhizdat, Moscow, 1953.
6. Hunter, Contact rolling problem of a stiff cylinder over a viscoelastic half-space. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 28, №4, 1961.

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